Jan 24, 2017

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Let us recall the concepts of base and local base for a topological space $(X, \mathcal{T})$.

Definition of a base $\mathcal{B}$:

$$\{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\} \text{ gives the topology } \mathcal{T}$$

Equivalent condition

$$\forall\, G \in \mathcal{T} \text{ and } \forall\, x \in G,\ \exists\, U \in \mathcal{B},\ x \in U \subset G$$

Definition of a local base $\mathcal{U}_x$ at $x \in X$

$$\forall\, \underbrace{\text{nbhd } N \text{ of } x}_{G \in \mathcal{T} \text{ with } x \in G},\ \exists\, U \in \mathcal{U}_x,\ x \in U \subset N$$

From the above, it is easy to see that

Fact 1. $\bigcup_{x \in X} \mathcal{U}_x$ is a base for $\mathcal{T}$

Fact 2. A base can easily give a local base $\{B \in \mathcal{B} : x \in B\}$ at $x \in X$.

Let us recall two versions of countability

2^{nd} countable: $\exists$ countable base for $(X, \mathcal{T})$

1^{st} countable: $\forall\, x \in X$, $\exists$ countable local base at x

Obviously, by Fact 2, $C_{II} \Rightarrow C_I$.

Consider $\mathbb{R}^n$, standard topology is C_{II}

$$\mathcal{B} = \{B(q, \tfrac{1}{k}) : 1 \le k \in \mathbb{Z},\ q \in \mathbb{Q}^n\} \text{ is a base}$$

Think about the proof and the essential steps.

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What is special about $\mathbb{Q}^n$ or $\mathbb{Q}$ in $\mathbb{R}$?

Archimedes Property in a way:

$$\forall\ r<s\in\mathbb{R},\ \exists\ q\in\mathbb{Q},\quad q\in(r,s).$$

Definition. A set $D\subset X$ is dense if $\overline{D}=X$.

Let us write down the logical statement for $x\in\overline{D}$, i.e., $\forall\ U\in\mathcal{T}$ with $x\in U$, $U\cap D\neq\emptyset$

$\exists\ d\in D,\ d\in U$

$$\overline{D}=X \Leftrightarrow \forall\ x\in X,\ x\in\overline{D}$$

When $x\in X$ is arbitrary (no restriction)

$$\emptyset\neq U\in\mathcal{T}$$

$$\overline{D}=X \Leftrightarrow \forall\ \emptyset\neq U\in\mathcal{T},\ \exists\ d\in D,\ d\in U$$

This is analogous $\quad r<s\in\mathbb{R} \qquad \mathbb{Q} \qquad (r,s)$

Definition. $(X,\mathcal{T})$ is separable if there is a countable dense set.

Question

$$\begin{tikzcd}
 & C_I \arrow[d, Rightarrow, "??(a)"] \\
C_{II} \arrow[ur, Rightarrow, "\text{Yes}"] \arrow[r, Rightarrow, "??(b)"'] & \text{separable}
\end{tikzcd}$$

The answer to (a) is No!

Simply make uncountably many clones of a C_I space

The answer to (b) is YES!

We need to construct a countable set $Q \subset X$ from a countable base

$$\mathcal{B} = \{B_k : 1 \le k \in \mathbb{Z}\}$$

Simply pick any point $x_k \in B_k$, $1 \le k \in \mathbb{Z}$ and form $Q = \{x_k : 1 \le k \in \mathbb{Z}\}$

It remains to prove $\overline{Q} = X$, i.e.,

take any $\phi \ne U \in \mathcal{J}$ and as $\mathcal{B}$ is a base

$$U = \text{a union of sets from } \mathcal{B}$$
$$= \bigcup_{j=1}^{\infty} B_{k_j}$$

$U \ne \phi$ so there must be $B_{k_j} \ne \phi$

$x_{k_j} \in B_{k_j} \subset U$, clearly $x_{k_j} \in Q$.

Question. Assume that X is C_I and separable.

$\therefore \exists$ countable $Q \subset X$ and $\overline{Q} = X$

Let $\mathcal{B} = \bigcup_{q \in Q} \mathcal{U}_q$.

It is certainly countable.

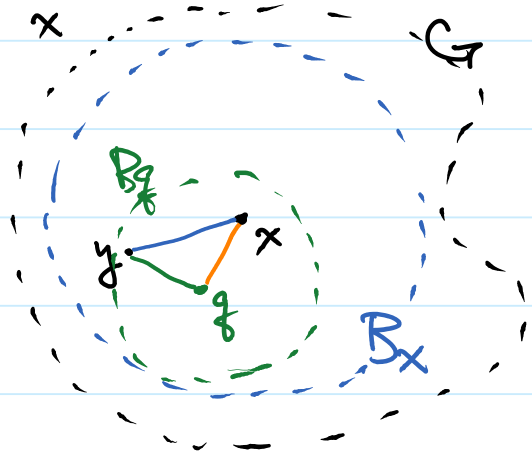
Would it be a base?

Let us revisit $\mathbb{Q}^n$ in $\mathbb{R}^n$.

For any open set $G \subset \mathbb{R}^n$ and $x \in G$

We have a ball B_x center at x

Need to insert another ball B_q center at $q \in \mathbb{Q}^n$



$$x \in B_q \subset B_x$$

$$\forall y \in B_q,\ y \in B_x$$

We need the argument

$$d(y,x) \le d(y,q) + d(q,x)$$

small by choice small by $\mathbb{Q}^n$ is dense

However, for other topology, we may not have Δ-inequality

Example. $\mathbb{R}$ with $\mathcal{T}_{ll}$ generated by $[a,b)$, $a<b \in \mathbb{R}$

* C_I : at $x \in \mathbb{R}$, $\mathcal{U}_x = \{ [x, x+\frac{1}{k}) : 1 \le k \in \mathbb{Z} \}$
* separable : we also have $\overline{\mathbb{Q}} = \mathbb{R}$ in $\mathcal{T}_{ll}$
* $\{ [p,q) : p,q \in \mathbb{Q} \}$ is not a base

Take $x \notin \mathbb{Q}$ and $x \in [x, x+\varepsilon) \in \mathcal{T}_{ll}$

we need to find $p,q \in \mathbb{Q}$ such that

$$x \in [p,q) \subset [x, x+\varepsilon)$$

$p \lneq x$ $x \lneq p$

Exercise. Use similar idea to show $\mathcal{T}_{ll}$ is not C_{II}

Consider mapping $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$

which respects/preserves certain topological properties

Example. For $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$, standard topology

$x_0\in X$ and $f(x_0)\in Y$

Think about the definition of continuity at x_0,

$\forall\, \varepsilon>0\ \exists\, \delta>0$ such that

if $\underbrace{\|x-x_0\|<\delta}_{d_X(x,x_0)<\delta}$ then $\underbrace{\|f(x)-f(x_0)\|<\varepsilon}_{d_Y(f(x),f(x_0))<\varepsilon}$

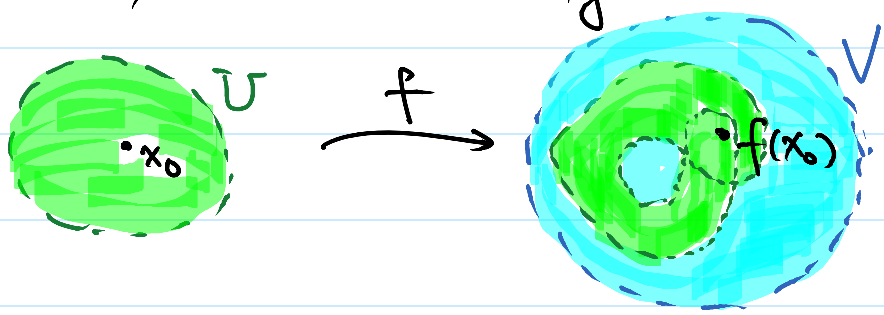
Can be rewritten as

if $x\in \underbrace{B_X(x_0,\delta)}_{U}$ then $f(x)\in\underbrace{B_Y(f(x_0),\varepsilon)}_{V}$

i.e., if $x\in U$ then $f(x)\in V$

The above can be translated to set language as $f(U)\subset V$ or $U\subset f^{-1}(V)$

In picture, this is exactly



What about $\forall\,\varepsilon>0,\ \exists\,\delta>0, \cdots$?

$\forall\,\varepsilon>0 \cdots B(f(x_0),\varepsilon)$ can be replaced with

$\forall\, V\in\mathcal{T}_Y$ where $f(x_0)\in V$

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Definition $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ is continuous at $x_0\in X$ if

$\forall\, V\in\mathcal{T}_Y$ with $f(x_0)\in V$

$\exists\, U\in\mathcal{T}_X$ with $x_0\in U$ such that

$f(U)\subset V$, equivalently, $U\subset f^{-1}(V)$

The mapping f is continuous everywhere if $x_0\in X$ becomes arbitrary.

In this case, $f(x_0)\in V$ really means $V\cap f(X)\neq\phi$

The statement becomes $\forall\, V\in\mathcal{T}_Y$ with $V\cap f(X)\neq\phi$

$$\underbrace{\forall\, x\in f^{-1}(V),\ \exists\, U\in\mathcal{T}_X,\ x\in U\subset f^{-1}(V)}_{\text{simply means } f^{-1}(V)\in\mathcal{T}_X}$$

Even if $V\cap f(X)=\phi$ then $f^{-1}(V)=\phi\in\mathcal{T}_X$

So, we arrive at a simple version.

Definition. A mapping $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ is continuous (everywhere) if

$\forall\, V\in\mathcal{T}_Y,\ f^{-1}(V)\in\mathcal{T}_X$